C-R equations

$f(z)$ is analytic $\Rightarrow f'(z)$ exists and is unique

$$\Rightarrow f'(z) = \underset{\delta z \to 0}{Lt} \frac{f(z+\delta z) - f(z)}{\delta z}$$

$\therefore \cancel{f'(z) = Lt}$

$$f(z) = U(x,y) + iV(x,y)$$

$$f(z+\delta z) = U(x+\delta x, y+\delta y) + iV(x+\delta x, y+\delta y)$$

$$\therefore f(z+\delta z) - f(z) = U(x+\delta x, y+\delta y) - U(x,y) + i\left[V(x+\delta x, y+\delta y) - V(x,y)\right]$$

$$\frac{f(z+\delta z) - f(z)}{\delta z} = \frac{U(x+\delta x, y+\delta y) - U(x,y)}{\delta x + i\delta y} + \frac{i\left[V(x+\delta x, y+\delta y) - V(x,y)\right]}{\delta x + i\delta y}$$

$$\therefore \underset{\delta z \to 0}{Lt} \Rightarrow \underset{\delta x \to 0,\ \delta y \to 0}{Lt}$$

Case I :- Let δz be purely real.

$$\therefore f'(z) = \underset{\delta x \to 0}{Lt} \frac{U(x+\delta x, y) - U(x,y)}{\delta x} + \frac{i\left[V(x+\delta x, y) - V(x,y)\right]}{\delta x}$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

Case II :→ Let δz be purely imaginary. ∴ $\delta x = 0$ and $\delta y \to$

$$\therefore f'(z) = \lim_{\delta y \to 0}\left\{\frac{u(x, y+\delta y) - u(x,y)}{i\,\delta y} + \frac{v(x, y+\delta y) - v(x,y)}{\delta y}\right.$$

$$= -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$\therefore \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

$$\therefore \boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}} \text{ and } \boxed{\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$$

⇒ continuity does not imply necessarily differentiability

$f(z) = |z|^2 = x^2 + y^2$ is cont

∴ ~~$f(z) - f(0)$~~ Now $f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$

$$\Rightarrow f'(z) = \lim_{\Delta z \to 0} \frac{(z+\Delta z)(\bar z + \overline{\Delta z}) - z\bar z}{\Delta z}$$

$$f'(z) = \underset{\Delta z \to 0}{Lt} \frac{f(z+\Delta z) - f(z)}{\delta z} = \underset{\Delta z \to 0}{Lt} \frac{(z+\Delta z)(\bar{z}+\overline{\delta z}) - z\bar{z}}{\Delta z}$$

$$= \underset{\Delta z \to 0}{Lt} \frac{\cancel{z\bar{z}} + \bar{z}\Delta z + z\overline{\delta z} + \Delta z \cdot \overline{\Delta z} - \cancel{z\bar{z}}}{\Delta z}$$

$$= \underset{\Delta z \to 0}{Lt} \left[\bar{z} + \frac{\overline{\Delta z}}{\Delta z} + \overline{\Delta z} \right]$$

$$= \bar{z} + \underset{\Delta z \to 0}{Lt} \frac{\overline{\Delta z}}{\Delta z} + \underset{\overline{\Delta z} \to 0}{Lt} \overline{\Delta z} \qquad \left|\ \because \Delta z \to 0 \Rightarrow \overline{\delta z} \to 0 \right]$$

$$= \bar{z} + \underset{\Delta z \to 0}{Lt} \frac{\overline{\Delta z}}{\Delta z} + 0.$$

$$= \bar{z} + \underset{\Delta z \to 0}{Lt} \frac{\overline{\Delta z}}{\Delta z}$$

Let $\Delta z = r(\cos\theta + i\sin\theta) = re^{i\theta}$

$\overline{\Delta z} = re^{-i\theta}$

$\therefore \dfrac{\overline{\delta z}}{\Delta z} = e^{-2i\theta}$ which depends on values of θ

$\therefore \underset{\Delta z \to 0}{Lt} \dfrac{\overline{\Delta z}}{\Delta z}$ has several value

$= \bar{z} +$